Unit 2

Interest and Time Value of Money

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- Introduction to Time Value of Money
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Time Value of Money

- The time value of money (TVM) assumes that money is worth more now than at a future date based on its *earning potential*.
- Money can *grow* when *invested*, any *delay* is a *lost opportunity for growth*.
- Money has time value because the *purchasing power* of money as well as the *earning power* of money changes with time.
- During *inflation*, *purchasing power* of money *decreases* over time.
- Money can earn an *interest* for a period of time. Interest represents the *earning power* of money.

Interest

- *Interest* refers to the cost of borrowing money or the return on investment for lending money.
- Interest is calculated as a *percentage of principal and calculated in a specific time frame.*
- Interest serves as an *incentive* for lenders to provide funds and *compensates* them for the *opportunity cost* of not using the money elsewhere.

Factors for Interest Calculation

1. Liquidity

- Liquidity is a measure of how easily cash can be obtained to pay short-term obligations or how quickly an asset can be converted into cash.
- Interest in this case is the reward for not being able to use money while holding the stock.

2. Risk Premium

Risk is inevitable (certain degree of risk is associated) in any financial investment.
Interest, in this case is the reward for any chance that the investor would not get money back (declined in value while invested).

3. Inflation Factor

Inflation is the rate of *increase* in *prices over a given period of time*.
Inflation leads to a *decrease* in the purchasing power of money.
Interest here is the *compensation* for *decrease in purchasing power* between the *time of investment* and the *time of return*.

Due to these factors, every investor expects some return on investment and charge a cost of investment known as Interest Rate.

Elements of Transaction Involving Interest

1. An original amount of money borrowed or invested is called *Principal*.
2. The *Interest Rate* measures the cost of money within a specific period of time, expressed in percentage.
3. Interest Period, a period of time that determines how frequently interest is calculated (weekly, monthly, quarterly etc.)
4. A specified length of time that makes the duration of transactions and thereby establishes a certain number of *interest periods*.
5. A future amount of money that results from the cumulative effects of interest rate over a number of interest periods.

Simple Interest

- Simple interest is an *interest charge* that borrowers pay lenders for a loan.
- Interest is calculated on *Principal only*.
- In simple interest, the principal amount is always the same.

Simple Interest (SI)= P*T*R/100

where,

P: Principal amount borrowed

T: Time duration for which the principal amount is borrowed

R: Rate of Interest at which principal is borrowed

Amount available at the end of T period is

A= P+SI= P(1+RT) *where R is in decimal.*

$$S.I. = \frac{P \times R \times T}{100}$$

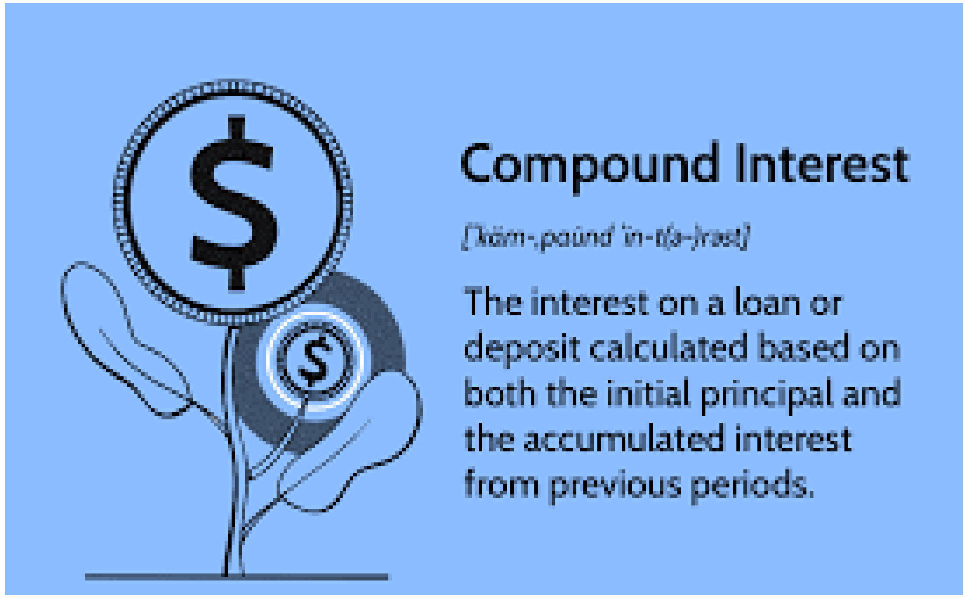
$$P = \frac{100 \times S.I.}{R \times T}$$

$$R = \frac{100 \times S.I.}{P \times T}$$

$$T = \frac{100 \times S.I.}{P \times R}$$

Compound Interest

- Compound interest is *interest calculated* on both the *initial principal* and *all of the previously accumulated interest*.
- It is usually termed as *interest over the interest*.
- Compound interest is calculated at *regular intervals* (monthly, annually etc.)
- Banks or any financial organization calculate the amount based on *compound interest only*.



Compound Interest…

$$CI = P\,(1 + r/n)^{nt} - P$$

where,

- ***P is the principal amount***
- ***r is the rate of interest(decimal obtained by dividing rate by 100)***
- ***n is the number of times the interest is compounded annually***
- ***t is the overall tenure.***

For an instance,

If the given principal is compounded annually, then we have n = 1 and in this case, the above formulas turn into the following:

Compound interest, $C.I = P(1 + r)^t - P$

Compound amount, $A = P(1 + r)^t$

Nominal Interest Rate

- Nominal interest rate refers to the interest rate before taking inflation into account.
- It can also refer to the advertised or stated interest rate on a loan, without taking into account any fees or compounding of interest.

Nominal Interest rate (NI)= r% compounded M-ly

where,

r: nominal rate per year

M: compounding frequency

Interest rate=r/M

For example: For example, a nominal annual interest rate of 12% based on monthly compounding means a 1% interest rate per month (compounded).

Effective Interest Rate

- The effective interest rate is the *overall interest rate* that an investor (or borrower) can get (or pay) in a year after the *compounding* is considered.
- The effective interest rate is the *usage rate* that a borrower actually pays on a loan, credit card, or any other debt amount.
- The effective interest rate is the *real interest return* on a savings account when the effects of compounding over time are taken into account.

$$r = (1 + i/n)^{n} - 1$$

where,

r = The effective interest rate

i = The stated interest rate

n = The number of compounding periods per year

Examples

A loan document contains a stated interest rate of 10% and mandates quarterly compounding. Find Effective Interest Rate on a loan document.

Solution:

The interest rate = 10%

The number of compounding periods per year = 4

Using effective interest rate formula,

$r = (1 + i/n)^n - 1$

$=(1 + 10\%/4)^4 - 1$

$=(1 + 0.10/4)^4 - 1$

$= 10.38\%$

Examples...

What is the effective interest rate for a nominal annual interest rate of 12% compounded monthly?

Solution:

Given:

The interest rate = 12%

The number of compounding periods per year = 12

Using the effective interest rate Formula,

$r = (1 + i/n)^{n} - 1$

$= (1 + 12\%/12)^{12} - 1$

$= (1 + 0.12/12)^{12} - 1$

= 12.68%

- The effective interest rate is higher than the nominal rate, unless the compounding frequency is on an annual basis.

Why?

The nominal interest rates neglect the effects of compounding, while the effective interest rates take into account the impact of compounding periods.

Continuous Compounding

- Continuously compounding interest means that an *account balance* is *constantly earning interest*, as well as *refeeding* that interest *back* into the balance so that it, too, *earns interest*.
- Continuous compounding *calculates* interest under the *assumption* that *interest will be compounding over an infinite number of periods*.
- It's *not possible* in the real world to *have an infinite number of periods* for interest to be *calculated* and *paid*.

If $\frac{M}{r} = p$,

$\frac{r}{M} = \frac{1}{p}$,

M= rp

$$\left(1+\frac{r}{M}\right)^{M}$$

$$= \left(1+\frac{1}{p}\right)^{rp}$$

$$= \left[\left(1+\frac{1}{p}\right)^{p}\right]^{r}$$

$$= e^{r}$$

$$i = \lim_{M\to\infty}\left(1+\frac{r}{M}\right)^{M} - 1 = \lim_{M\to\infty}\left[\left(1+\frac{r}{M}\right)^{\frac{M}{r}}\right]^{r} - 1 = e^{r} - 1$$

$$i = e^{r} - 1$$

$$e^{r} = 1 + i$$

$$e^{rN} = (1+i)^{N}$$

where,
M= number of compounding period per year
r= Interest Rate
P= Principal
N= Number of years

Continuous Compounding formula

1. For Discrete Cash flows

- Discrete cash flows assume the cash flows occur at a discrete intervals (e.g. once a year), but continuous compounding assumes compounding is continuous throughout the interval.($M = \infty$)

Substitute $e^r = 1 + i$

$F = A (e^{rN}-1)/(e^r-1)$

$P = A (e^{rN}-1)/((e^r-1)*e^{rN})$

r% denotes nominal rate continuous compounding

2. For Continuous Cash Flows

- A sequence of cash flow that occur in incredibly short intervals of time is referred as continuous cash flow.
- It may have annuity having an infinite number of short time.

$F = \hat{A}*(e^{rN}-1)/r$ $F = \hat{A} * (F/\hat{A}, \underline{r}\%, N)$

$P = \hat{A}*(e^{rN}-1)/(r*e^{rN})$ $P = \hat{A} * (P/\hat{A}, \underline{r}\%, N)$

For Discrete Cash Flow

What will be FW at the end of 5 years of cash flow at the rate of Rs. 500 per year for 5 years with interest compounded continuously at nominal annual rate of 8%.

F = A * (F/ A, r%, N) F = A * (F/A, 8%, 5)

F = A $(e^{rN}-1)/(e^{r}-1)$

F = 500 $(e^{0.08*5}-1)/(e^{0.08}-1)$ =

F= Rs. 2952.58

For Continuous Cash Flow

What will be FW at the end of 5 years of cash flow at the rate of Rs. 500 per year for 5 years with interest compounded continuously at nominal annual rate of 8%.

F = Â * (F/ Â, r%, N) F = Â * (F/ Â, 8%, 5)

F = Â*$(e^{rN}-1)/r$

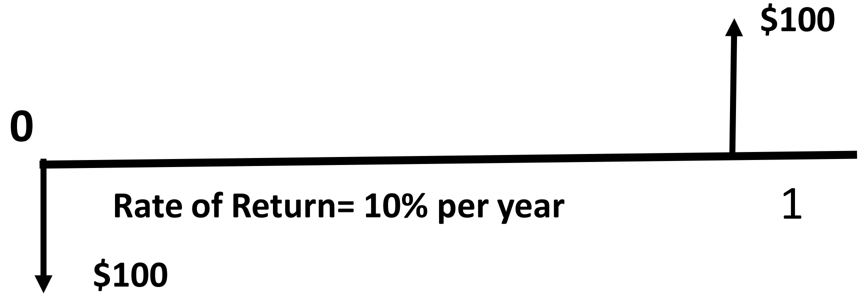
F = 500 $(e^{0.08*5}-1)/0.08$

F= Rs. 3074.00

Economic Equivalence

- Economic equivalence is a combination of *interest rate* and *time value of money* to determine the *different amounts of money* at *different points in time* that are *equal in economic value*.
- Economic equivalence is a *fundamental concept* upon which engineering economy computations are based.
- For an instance, \$100 now is economically equivalent to \$110 one year from now, if the \$100 is invested at a rate of 10% per year.

$$A = P(1+r*t) = 100\,(1+0.1*1) = \$110$$



Example

Assume an engineering company borrows $100,000 at 10% per year compound interest and will pay the principal and all the interest after 3 years. Compute the annual interest and total amount due after 3 years.

To include compounding of interest, the annual interest and total owed each year are calculated by

Interest, year 1: 100,000(0.10) = $10,000

Total due, year 1: 100,000 + 10,000 = $110,000

Interest, year 2: 110,000(0.10) = $11,000

Total due, year 2: 110,000 + 11,000 = $121,000

Interest, year 3: 121,000(0.10) = $12,100

Total due, year 3: 121,000 + 12,100 = $133,100

Example :

You borrowed \$5,000 from a bank and you have to pay it back in 5 years with an annual interest of 8%.

What are the possible ways to repay the debt?

Options to repay

Plan 1: At end of each year pay $1,000 principal plus interest due.

Plan 2: Pay interest due at end of each year and principal at end of five years.

Plan 3: Pay in five end-of-year payments.

Plan 4: Pay principal and interest in one payment at end of five years.

All these plans are equivalent in the sense that the sum of all outgoing cash flows at time 0 is $5,000

General Principal

Principle 1: Cash flow comparisons require a common time basis.

- Alternative cash flow scenarios must be compared within the same time period. This means we need to “move” cash flows to put them all in the same time period.

In example, we compared the two cash flows in the same time period as per principle 1. We applied interest to $100 in the current period to obtain the future value after one year using the future value formula introduced in the Interest section. This yields the future value of $110. As principle 1 states, we have to find the future value of the present cash flow for it to be comparable to the other cash flow that occurs one year from now. We then compared the two cash flows, arriving to conclusion that the two are economically equivalent.

Principal 2: Economic equivalence is dependent on the "discount rate".

To account for the time value of money while "moving" these cash flows we must *reduce* the economic value of future cash flows. Why? Recall that a dollar today is worth more than a dollar tomorrow, which means that a dollar tomorrow is worth *less* than a dollar today. So, when converting future cash flows to their equivalent (lower) present value to compare these cash flows we use what is termed a discount rate.

A "discount rate" is the rate used to discount future cash flows to adjust for the time value of money.

The discount rate is higher than the interest rate, as it accounts for other costs as well, such as opportunity costs

Principle 3: Comparisons may require conversion to a single cash flow.

When there are multiple cash flows to be considered in an alternative cash flow scenario, the individual cash flows may need to be combined into a single cash flow for comparison.

Example: Say Ishan now has to decide whether to receive $100 from Rahul now or, as an alternative, $60 in the first year and $50 in the second year, as shown in the diagram below. Like before, he has access to a deposit account that pays 10% compounded annually.

Principle 4: Equivalence is maintained regardless of point of view.

As long as the same discount rate is used, the equivalence between cash flows will be maintained at any point in time to any party involved in the transactions. That is, it doesn't matter if you are paying or receiving money. Economic equivalence will be the same from both perspectives.

Let's analyze the cash flow diagrams from Ishan's and Rahul's point of view. Comparing $100 today and $110 a year from now with 10% interest compounded annually, as per principle four, the economic equivalence will be maintained for both Ishan and Rahul.

Development Interest Formula

1. Single Cash Flow

- Single cash flows involve a single financial transaction at a point in time.
- A present sum P invested now for N interest periods at interest rate i% per period, its future worth F would be $F = P * (1 + i)^N$
- The future value formula enables us to “move” cash flows to a future point in time.
- The factor $(1 + i)^N$ is termed as Single Payment Compound Amount Factor.
- The interest rate i is also known as discount rate and the P/F, i, N factor is termed as discounting factor.

- *You are an employee at ABC Company. Your company offers you two options for a salary bonus: you can either accept a $1000 bonus now, or you can wait and take a $1200 bonus two years from now. Which one should you choose if your discount rate is 5% per year?*

Solution,

Condition 1: Discount the $1200 bonus for two interest periods (two years) at 5% annually to obtain its present value.

Here, N=2 and F= $1200

$P = F/(1+i)^N = 1200/(1+0.05)^2$

$= \$1088.44$

Condition 2: At a 5% discount rate, the $1200 bonus has a present value greater than $1000.

In this condition, the $1200 bonus should be chosen.

2. Uniform Series

- A uniform series, sometimes called an equal-payment series, is a cash flow series in which the same amount of money is paid or received in two or more sequential periods.
- These types of equal payments are also referred to as annuities.
- This type of cash flow series is probably familiar as many long-term loans, such as house mortgages and car loans, involve a fixed monthly payment for a set length of time.
- If an amount A is invested at the end of each periods for N interest periods at interest rate i% per period, its future worth F would be

$$F = A\,\frac{(1+i)^N - 1}{i}$$

To calculate the present value, *P*, from the regular payment

$$P = A[\frac{(1+i)^N - 1}{i(1+i)^N}]$$

- *N* : the number of periods in the uniform series,
- *i* : the interest (or discount rate) per period, and
- *A* : the equal cash flow per period.
- N, i, and A must all be in the same time frame.

Note: If an example requires calculating a monthly payment (*A*), then *N* must be months and *i* must be the monthly interest rate.

Examples

1. *Robin wants to borrow $12,000 and pay it off in equal monthly payments over a 5-year period. If the monthly interest rate is 0.75%, compounded monthly, how much would each payment be?*

Given,

N = 5 years or 60 months,
i = 0.75% = 0.0075, and
P = $12,000

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$$

Monthly payment (A)= $249.10

Total amount paid= $249.10*60 = $14946

Total interest paid= $14946-$12000= $2946

2. If you wish to withdraw Rs. 10,000 at the end of each year at an interest rate of 10% per annum for 4 years. How much amount should you deposit now?

A= Rs 10,000

i= 10%

N= 4 years

$$P = A[\frac{(1+i)^N - 1}{i(1+i)^N}]$$

P =10000*((1+0.1)^4-1)/(0.1*(1+0.1)^4)

P= Rs 31698.95

“In engineering, every design choice has a cost; great engineers balance innovation with financial wisdom”


THANK
YOU

Unit 2

Interest and Time Value of Money

Development Interest Formula contd…

3. Uneven Payment Series

- Uneven payment series is a series of cash flow that are not t equal.
- There is no single formula available to compute the present or future value of a series of uneven cash flows.

Present Value

- When we have unequal cash flows, we must first find the present value of each individual cash flow and then the sum of the respective present values.

Future Value

- Once we know the present value of the cash flows, we can easily apply time-value equivalence by using the formula to calculate the future value of a single sum of money

DISCRETE CASH FLOWS AND DISCRETE COMPOUNDING			
To Find:	Given:	Factor	Factor Name
For single cash flows:			
F	P	$(1 + i)^N$	Single Payment Compound Amount Factor
P	F	$(1 + i)^{-N}$	Single Payment Present Worth Factor

- When calculating the PV of an uneven cash flow stream, it should always be less than the sum of the cash flows.
- When calculating the FV of an uneven cash flow stream, it should always be more than the sum of the cash flows.
- $P = F * (1 + i)^{-N}$ to calculate present value when future value is given
- $F= P* (1+i)^N$ to calculate future value when present value is given

Example

- John wants to pay off his student loan in three annual installments: \$2,000, \$4,000 and \$6,000, respectively, in the next three years. How much should John deposit into his bank account today if he wants to use the account balance to pay off the loan? Assume that the bank pays 8% interest, compounded annually.

$P= F/(1+i)^N$

P1= 2000/(1+0.08)^1= \$1852 (For year 1)

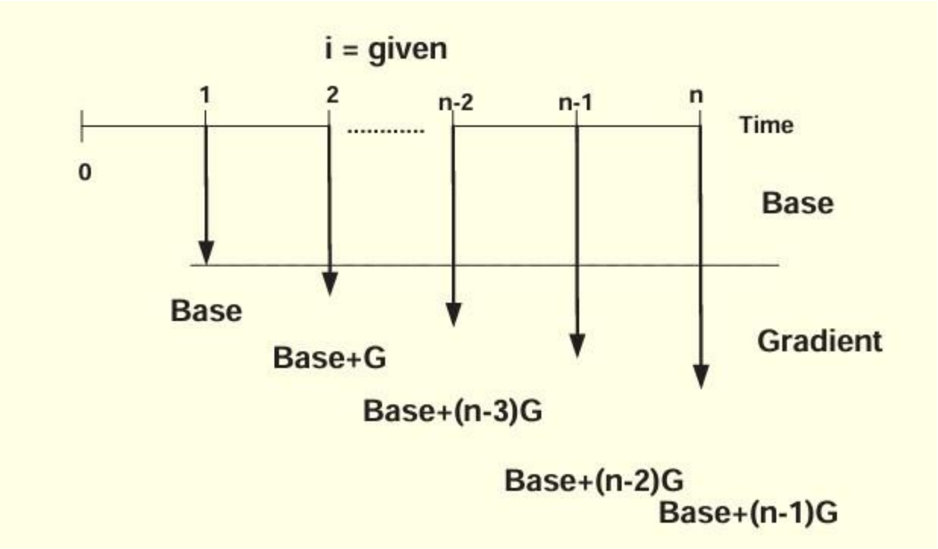
P2= 4000/(1+0.08)^2= \$3429 (For year 2)

P3= 6000/(1+0.08)^3 = \$4763 (For year 3)

Total amount to be deposited = \$ 10,044

Linear Gradient Series

- A linear gradient series is a series of cash flows which increase or decrease by a constant amount every period.
- The cash flow changes by the same amount each period.
- The amount of decrease or the increase is the gradient or G.
- Therefore, it is composed of base amount and gradient part.
- The first cash flow in the gradient is series is 0 (ie, when N=1, G=0)



- If an amount **A** is invested or paid at the end of interest period 1 changes (increases or decreases) by a constant amount **G** at the end of each periods for N interest periods at interest rate i% per period,

$$F = \frac{G}{i}\left[\left\{\frac{(1+i)^N - 1}{i} - N\right\}\right]$$

$$F = \frac{G}{i} * \{(F/A, i, N) - N\}$$

$$F = \frac{G}{i} * (F/A, i, N) - \frac{NG}{i}$$

$$F = G * (F/G, i, N)$$

The factor (F/G, i, N) is termed as gradient series compound amount factor.

For uniform linear gradient, G starts from at the end of 2nd period onwards at the end of each period for N periods with i% interest period.

$$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right] = G(P/G, i, N)$$

$F = P(1+i)^N$

$$F = A\frac{(1+i)^N-1}{i}$$

$$P = A\frac{(1+i)^N-1}{i(1+i)^N}$$

$$F = \frac{G}{i}\left[\left\{\frac{(1+i)^N-1}{i} - N\right\}\right]$$

$F = P(1+i)^N$	F = P * (F/P, i, N)
$F = A\frac{(1+i)^N-1}{i}$	F = A * (F/A, i, N)
$F = G\left[\frac{1}{i}\left\{\frac{(1+i)^N-1}{i} - N\right\}\right]$	F = G *(F/G, i, N)

Numerical

A textile mill has just purchased a lift truck that has a useful life of five years. The engineer estimates that maintenance costs for the truck during the first year will be $1,000. As the truck ages, maintenance costs are expected to increase at a rate of $250 per year over the remaining life. Assume that the maintenance costs occur at the end of each year. The firm wants to set up a maintenance account that earns 12% annual interest. All future maintenance expenses will be paid out of this account. How much does the firm have to deposit in the account now?

Given,

A= \$1000

G= \$250

N= 5 years

i= 12% per annum

First calculate P for Base Amount,

$$P = A \frac{(1+i)^N - 1}{i(1+i)^N}$$

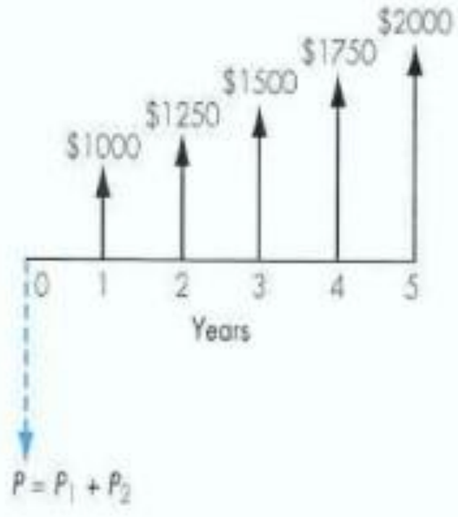
P1 = \$3604.8

Secondly, calculate P for Gradient Amount,

$$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right] = G(P/G, i, N)$$

P2= \$ 1599.3

Total amount to be deposited in the account (P) = P1+P2= \$5204.03



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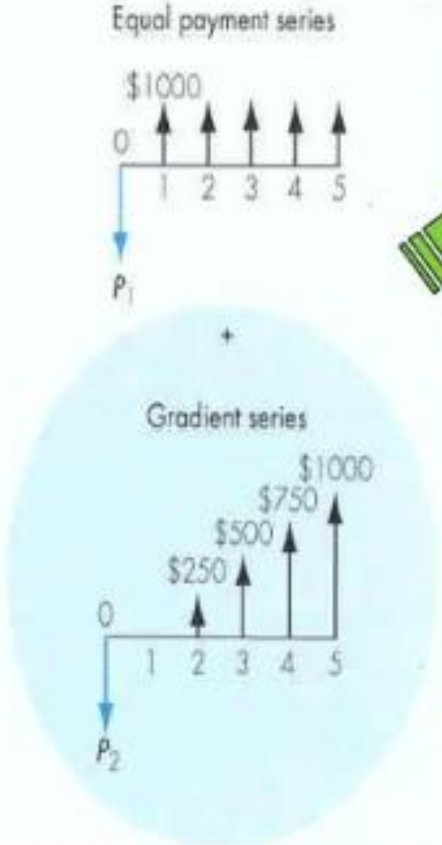


$$P_1 = \$1{,}000(P/A,12\%,5)$$
$$= \$3{,}604.80$$

$$P_2 = \$250(P/G,12\%,5)$$
$$= \$1{,}599.20$$

$$P = \$3{,}604.08 + \$1{,}599.20$$
$$= \$5{,}204$$

John and Barbara have just opened two savings accounts at their credit union. The accounts earn 10% annual interest. John wants to deposit $1,000 in his account at the end of the first year and increase this amount by $300 for each of the next five years. Barbara wants to deposit an equal amount each year for the next six years. What should be the size of Barbara’s annual deposit so that the two accounts will have equal balances at the end of six years?

- Amount to be deposited by John (A1)= $1000
- Gradient amount (G)= $300
- N= 6 years
- i= 10%
- To calculate Future Gradient Amount using $F = G\left[\frac{1}{i}\left\{\frac{(1+i)^N - 1}{i} - N\right\}\right]$

F= $667.06

Barbara’s total annual deposit Amount = A1+F

= $1000+$667.06 = $1667.06

Three counties in Florida agreed to pool tax resources already designated for county-maintained bridge refurbishment. At a recent meeting, the county engineers estimated that a total of $500,000 will be deposited at the end of next year into an account for the repair of old bridges throughout the three-county area. Further, they estimate the deposits will increase by $100,000 per year for only 9 years thereafter, then cease. Determine the equivalent present worth if county funds earn interest at a rate of 5% per year.

Given,

Estimated total amount (A)= $500,000
Gradient (G) = $100,000
Interest (i) = 5%
N= 9 years + end of next year= 10 years

$$
\begin{aligned}
P_T &= 500\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] + 100\frac{1}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right] \\
&= 500\left[\frac{(1.05)^{10} - 1}{0.05(1.05)^{10}}\right] + 100\frac{1}{0.05}\left[\frac{(1.05)^{10} - 1}{0.05(1.05)^{10}} - \frac{10}{(1.05)^{10}}\right] \\
&= 500(7.7217) + 100(31.6520) = 7026,05 => \$7,026,050
\end{aligned}
$$

Geometric Gradient Series

- A geometric gradient series is a cash flow series that either increases or decreases by a constant percentage each period.
- It is common for annual revenues and annual costs such as maintenance, operations, and labor to go up or down by a constant percentage, for example, +5% or -3% per year.
- The uniform change is called the rate of change.
- The (P/A, g, i, n) factor calculates P_g in period $t = 0$ for a geometric gradient series starting in period 1 in the amount A_1 and increasing by a constant rate of g each period.

- The Present Worth (P) is given by:

$$P_g = A_1 \left[\frac{1 - \left(\frac{1+g}{1+i}\right)^n}{i - g} \right] ; \quad g \neq i \qquad\qquad P_g = \frac{nA_1}{(1+i)} \quad ; g=i$$

= A_1 (P/ A_1, g, i, n)

Where, g = gradient percent (as a decimal in calculations)

i = interest percent (as a decimal in calculations)

n = number of periods

A_1 = payment at EOY 1

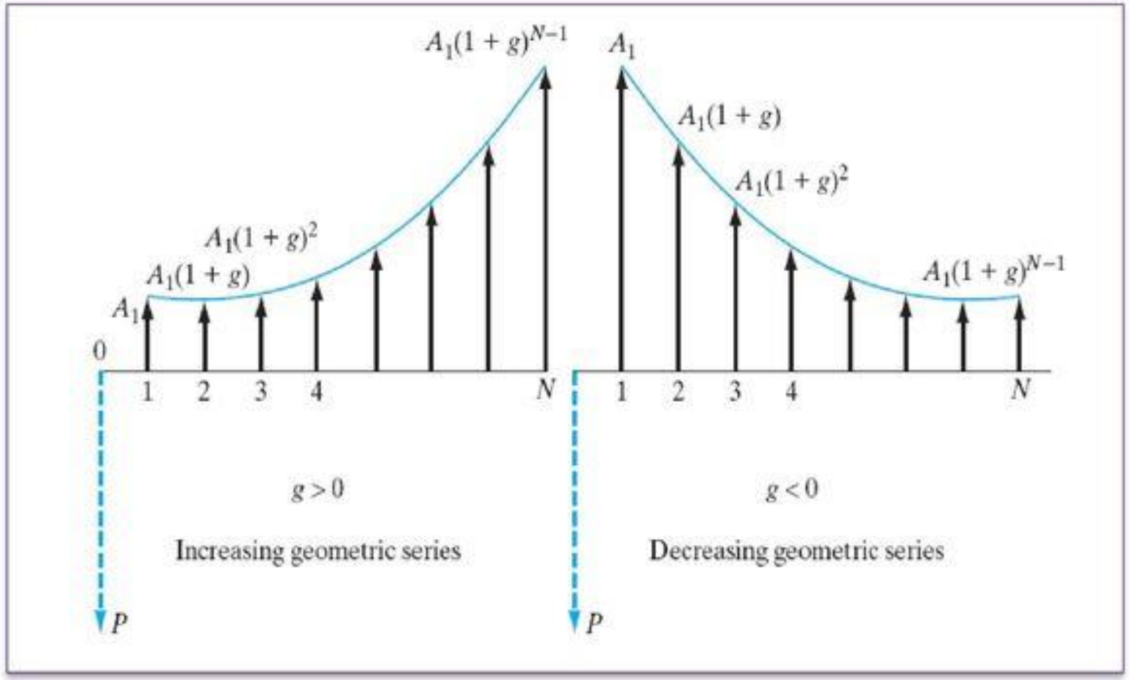
Present Worth of Geometric Gradient Series

❑Formula

$$P = \begin{cases} \dfrac{A_1}{i-g}\left[1 - \left(\dfrac{1+g}{1+i}\right)^N\right] & \text{if } i \neq g \\ \dfrac{NA_1}{1+i} & \text{if } i = g \end{cases}$$

❑Factor Notation

$$P = A_1(P/A_1, g, i, N)$$



To calculate future value, after P is known

- Once P is known,

$F = P(1+i)^N$

For i≠g,

$$F = \frac{A_1}{(i-g)}\left[(1+i)^N - (1+g)^N\right]$$

For i=g,

$$F = N\frac{A_1}{(1+i)}(1+i)^N$$

Examples

1. If fuel consumption at the end of year one is \$2000 & increases at 8% per year thereafter for next three years. What is its equivalent P & F at i= 5%?

Given,

A1= \$2000

g= 8%

i= 5%

N= 4 years

P= \$7952

$$P_g = A_1 \left[\frac{1 - \left(\frac{1+g}{1+i}\right)^n}{i - g} \right] ; \quad g \neq i$$

2. A coal-fired power plant has upgraded an emission control valve. The modification costs only $8000 and is expected to last 6 years with a $200 salvage value. The maintenance cost is expected to be high at $1700 the first year, increasing by 11% per year thereafter. Determine the equivalent present worth of the modification and maintenance cost.

To calculate equivalent present worth (PT)= -A-Pg+200(P/F,8%,6)

= -8000-1700*(5.9559)+126

= $-17,999

Irregular or Mixed Series

- A mixed series refers to a cash flow pattern that is a combination of different types of cash flows, rather than a single uniform series.
- It may include a mix of lump sum payments, annuities (equal periodic payments), and varying cash flows at different time intervals.
- Add up all the present values to get the total present value of the mixed series.

How to Deal with Mixed Series

To calculate the present value (PV) or future value (FV) of a mixed series:

1. Separate Components: Break the cash flows into manageable parts (lump sums, annuities, and varying payments).

2. Discount or Compound Individually: Use the appropriate TVM formula for each part.

- For lump sums: $PV=FV/(1+r)^n$
- $FV=PV*(1+r)^n$
- For annuities: Use the annuity formulas for PV or FV.
- For uneven payments: Discount or compound each payment individually.

3. Sum the Results: Add up the present values or future values of all components.